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Abstract: We consider the Cauchy issue of recovering boundary values on the inner circle of a two-dimensional annulus from available overdetermined data on the outer circle, for solutions to the Laplace equation. Using tools from complex analysis, Hardy classes and approximation, we establish stability properties and error estimates, as well as a recovery scheme, illustrated by some numerical computations.

Key-words: inverse Cauchy problems, Hardy classes on annulus, bounded extremal problems

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Extensions analytiques et problèmes inverses de Cauchy sur une couronne. I : aspects théoriques et résultats de stabilité

Résumé : On considère un problème de Cauchy visant à retrouver les valeurs sur le cercle intèrieur d'une couronne bi-dimensionnelle d'une fonction harmonique dans la couronne, sur-déterminée sur le cercle extérieur. On établit des popriétés de stabilité et des estimations de l'erreur, en utilisant des outils d'analyse complexe, ainsi qu'une procédure de résolution par approximation dans les classes de Hardy, illustrée par quelques simulations numériques.

Mots-clés : problèmes inverses, problèmes de Cauchy, classes de Hardy de la couronne, problèmes extrémaux bornés

1 Introduction

1.1 Motivation

Among data extension issues in elliptic inverse problems there arises the task of recovering either Dirichlet (or Neumann) boundary data (temperature field, electric potential, ...), or a Robin type exchange coefficient, or a crack located on some interface of the structure, from overdetermined measurements on the outer boundary. In the case of a tube or pipe, the problem should reduce to a 2D one on an annulus.

This can also be related to the inverse electroencephalography problem in spherical 3D domains, which give simple models of the human head, assumed to be a ball made of (at least) three concentric spherical homogeneous layers (brain, skull, and scalp) [23, 8]. Overdetermined electrical measurements (potential, current flux) are available on the scalp (external boundary), from which one wants to recover some current sources (conductivity defaults) located in the brain (inner layer), the potential being harmonic elsewhere in the domain. Taking planar cross-sections of the ball allows one to express the problem in a family of discs, where the above 2D data extension issue may arise as a preliminary step before recovering the singularities.

These issues, and some others, usually involve solving a Cauchy problem for the Laplace operator on an annulus, from available data on the other boundary. This problem is known to be ill-posed since the work of Hadamard, its most critical feature being the lack of continuity of the solution – whenever it exists. (This is the case for compatible data, which means that the overdetermined data is indeed the trace and normal derivative of the solution of a single harmonic function.) Therefore great care is required when solving such a problem. There are several general algorithms for solving these problems or dealing with applications close to those intended in this work [6, 20, 27]; however all these algorithms solve effectively the Cauchy problem (i.e., we have the solution in the entire domain). They are based on multiple resolution of the backward problem and are therefore time consuming. Our approach is cheap and based on harmonic approximation. In fact, we mainly consider the issue of recovering a Robin coefficient on the inner circle of the annulus from overdetermined data on the outer boundary, both from a constructive viewpoint and by establishing stability results and error estimates. This can also be used in order to recover the (Dirichlet or Neumann) boundary data on the inner boundary. It is possible to analyse the same problems in sufficiently smooth domains that are conformally equivalent to an annulus, but it is convenient to begin by working directly in the annulus.

More precisely, let $\mathbb{D}$ be the unit disc and G be the annulus $G = \mathbb{D} \setminus \overline{s\mathbb{D}}$ for some fixed s with $0 < s < 1$. Consider the following inverse problem: given two functions u_b and Φ, or a number of their pointwise measurements, with $\Phi \not\equiv 0$, find a function φ, such that a solution u to

$$\begin{cases} \Delta u &= 0 \quad \text{in } G \\ u &= u_b \quad \text{on } \mathbb{T} \\ \partial_n u &= \Phi \quad \text{on } \mathbb{T} \end{cases} \tag{1}$$

also satisfies

$$\partial_n u + \varphi u = 0 \text{ on } s\,\mathbb{T}\,, \tag{2}$$

where ∂_n stands for the partial derivative w.r.t. the outer normal unit vector to $\mathbb{T}$. In the thermal framework, u_b and Φ correspond to the measured temperature and to the imposed heat flux on the outer boundary of some plane section of a tube, while φ is the exchange Robin coefficient to be recovered on the associated inner boundary.

In the present 2D situation of an annulus, as well as in conformally equivalent domains [25], one can make use of best approximation schemes in Hardy classes of analytic functions of the complex variable, in order to build an extension (in the annulus, up to its internal boundary) of the available data (on the outer boundary). This approach has already been successfully used, for the simply-connected case of the disc, in order to recover an unknown exchange coefficient on a part of the circle, from available data on the complementary part [15]. Here, it is constructively extended to domains bounded by two Jordan curves. A few preliminary numerical illustrations are provided in Section 4, for a number of potential applications. Computational issues, robustness and convergence properties of the algorithm will be more deeply studied in a companion paper [26].

Stability results and error estimates for the inverse Robin problem (with suitable norms) are established as consequences of boundedness properties for functions of weighted Hardy classes. This can be viewed as an extension of the results established in [2, 13], which hold on connected boundaries.

The overview of the present article is as follows. The next section, Section 1.2, is devoted to notation and preliminary well-posedness results. Using harmonic conjugation, we then introduce in Section 1.3 analytic functions associated to the problem, which in fact belong to Hardy spaces of an annulus, defined in Section 1.4. This allows us to express the problem in Section 1.5 in terms of recovery of functions in these Hardy classes from their trace on a subset of the boundary.

Extension / approximation schemes are then established in Section 2.1, for weighted Hardy spaces of an annulus, while boundedness properties are discussed in Section 2.2. Stability results and errors estimates for the inverse problem are then derived in Section 3. A few numerical illustrations are given in Section 4 (related numerical issues will be discussed and illustrated with more details in a companion paper [26]) and a conclusion is given in Section 5.

1.2 Notation and preliminary results

Let $G \subset \mathbb{R}^2 \simeq \mathbb{C}$ be a bounded domain, with Lebesgue measure ν. We let $L^2(G)$ be the Hilbert space of square-integrable functions on G (w.r.t. ν) while, for $r \in \mathbb{R}_+$, the Sobolev spaces $W^{r,2}(G)$ (of real or more generally complex valued functions) are defined as usual by [12, 19] :

$$W^{r,2}(G) = \{ f \in L^2(G) \,,\ \| f \|^2_{W^{r,2}(G)} = \int_G (1+|\xi|^2)^r \, |\tilde{f}(\xi)|^2 \, d\nu(\xi) \ < \infty \},$$

where $\tilde{f}$ is the Fourier transform of f. They become Hilbert spaces with the related inner product. Whenever $r = m \in \mathbb{N}$, it holds that

$$W^{m,2}(G) = \{ f \in L^2(G) \,,\ \| f \|^2_{W^{m,2}(G)} = \sum_{0 \le |p| \le m} \int_G | D^p f(\xi) |^2 \ d\nu(\xi) \ < \infty \} \,,$$

with, as usual,

$$|p| = p_1 + p_2 \,,\ D^p f = \frac{\partial^{p_1+p_2}}{\partial x_1^{p_1} \, \partial x_2^{p_2}} f \,.$$

Whenever ∂G is smooth enough ($C^{n+1,\beta}$, say, $0 < \beta < 1$), the following characterization of Sobolev spaces $W^{n+1/2,2}(\partial G)$ also holds, for $n \in \mathbb{N}$:

$$W^{n+1/2,2}(\partial G) = \{ f \in L^2(\partial G) \text{ s.t. } \exists \, F \in W^{n+1,2}(G) \ : \ F_{|\partial G} = f \},$$

with the norm $\| f \|_{W^{n+1/2,2}(\partial G)} = \inf\{ \| F \|_{W^{n+1,2}(G)} \,,\ F_{|\partial G} = f \}$. Further, there exist constants $k_{n,G}$ and $K_{n,G}$ such that for all $f \in W^{n+1,2}(G)$ we have:

$$k_{n,G} \, \|f_{|\partial G}\|_{W^{n,2}(\partial G)} \le \|f_{|\partial G}\|_{W^{n+1/2,2}(\partial G)} \le K_{n,G} \, \|f\|_{W^{n+1,2}(G)} \,.$$

Returning to the annulus $G = \mathbb{D} \setminus \overline{s\mathbb{D}}$, for some fixed s, $0 < s < 1$, we have the following existence and regularity result, concerning the associated Neumann to Dirichlet direct problem of finding the solution u and its trace u_b on $\mathbb{T}$ when Φ and φ are given in (1), (2).

Let c_s , $C_s > 0$ and introduce the following class of "admissible" Robin coefficients:

$$A^{(n)} = A^{(n)}(s,n,c_s,C_s) = \{\varphi \in C^n(s\,\mathbb{T}) \,,\ |\varphi^{(k)}| \le C_s,\ 0 \le k \le n,\ \text{and}\ \varphi \ge c_s \} \,.$$

Theorem 1 ([14], [15]) *Let $n \ge 0$, $\Phi \in W^{n,2}(\mathbb{T})$, $\Phi \ge 0$, $\Phi \not\equiv 0$ and assume that $\varphi \in A^{(n)}$, for some constants c_s , $C_s > 0$. Then there exists a unique function $u \in W^{n+3/2,2}(G)$, hence also $u_b \in W^{n+1,2}(\partial G)$, which is a solution to (1), (2).*

Further, there exists constants $m > 0$ and $\kappa > 0$ (depending on the class $A^{(n)}$) such that for all $\varphi \in A^{(n)}$ and $\Phi \in W^{n,2}(\mathbb{T})$,

$$u \ge m > 0 \ \text{on}\ s\mathbb{T} \,, \tag{3}$$

and

$$\|u\|_{W^{n+1,2}(\partial G)} \le \kappa \,. \tag{4}$$

The proofs of the above results, see [14, Lem. 2], [15, Thm 2], rely on shift and Sobolev embedding theorems, together with the Hopf maximum principle [19, 24, 31].

Remark 2 The prior assumptions on the problem above are mainly of two kinds. The first concerns the unknown Robin coefficient φ, which has to be smooth and bounded from below and above. The smoothness requirement is indeed a restrictive condition, technically needed for the inverse problem to make sense (see Theorem 3). Boundedness, however, corresponds to physical limitations on the exchange process on $s\mathbb{T}$, which in particular may not turn out to be perfectly insulating or conducting. The second set of hypotheses concern the imposed flux Φ on $\mathbb{T}$, which should be smooth enough and without change of sign (in order to guarantee that the solution does not vanish in G); although these are additional physical restrictions, they can be guaranteed to hold on $\mathbb{T}$, where Φ is chosen.

The next identifiability property [14, Thm 1] ensures the uniqueness of solutions φ to the inverse problem, which is a necessary prerequisite for the stability issue to make sense.

Theorem 3 ([14]) *Let $\Phi \in L^2(\mathbb{T})$, $\Phi \geq 0$, $\Phi \not\equiv 0$ and $\varphi_1, \varphi_2 \in A^{(0)}$. Let u_1 and u_2 be the associated solutions. If $u_{1|\mathcal{K}} = u_{2|\mathcal{K}}$ on some open subset $\mathcal{K} \neq \emptyset$ of $\mathbb{T}$, then $\varphi_1 = \varphi_2$.*

Note that in fact versions of Theorems 1 and 3 will hold also in $C^{n,\beta}$ (hence in C^{n+1}) smooth domains G.

1.3 Harmonic conjugation

Let $G = \mathbb{D} \setminus \overline{s\mathbb{D}} \subset \mathbb{C} \simeq \mathbb{R}^2$, $0 < s < 1$, be an annulus, its boundary $\partial G = s\mathbb{T} \cup \mathbb{T}$ being equipped with the Lebesgue measure normalized so that the circles $\mathbb{T}$ and $s\mathbb{T}$ each have unit measure.

Let $\Phi \in L^2(\mathbb{T})$ and assume that $\varphi \in A^{(0)}$. From Theorem 1, $u_{|\partial G} \in W^{1,2}(\partial G)$. There exists a locally single-valued function v harmonic in G such that $\partial_\theta \, v = \partial_n \, u$ on ∂G, where ∂_θ stands for the tangential partial derivative on ∂G, from the Cauchy–Riemann equations.

Note that v is given on $\mathbb{T}$ up to a constant by

$$v_{|\mathbb{T}}(e^{i\theta}) = \int_{\theta_0}^{\theta} \Phi(e^{i\tau}) \, d\tau \,,$$

for an arbitrary $e^{i\theta_0} \in \mathbb{T}$, a quantity which is available from (1). Thus, $f = u + i\, v$ is analytic (and many-valued) in G; it is given on $\mathbb{T}$ by

$$f(e^{i\theta}) = u_b(e^{i\theta}) + i \int_{\theta_0}^{\theta} \Phi(e^{i\tau}) \, d\tau \,. \tag{5}$$

Also, on $s\mathbb{T}$,

$$\varphi = -\frac{\partial_\theta \, v}{u} = -\frac{\partial_\theta \, \mathrm{Im}\, f}{\mathrm{Re}\, f} \,, \tag{6}$$

which gives the link to be used between φ and f, in order to recover φ from approximants to f on the outer part $\mathbb{T}$ of the boundary ∂G or to establish stability results as continuity properties of the map $(u\,,\;\Phi) \to \varphi$ (see Sections 3, 4).

However, since the annulus is not simply-connected, it may not be possible to define f globally in G as a single-valued function. Indeed, one can see from Green's formula applied to the solution u of (1), (2) and to any constant function in G, that

$$\int_{\partial G} \partial_n\, u = \int_{\mathbb{T}} \Phi - \int_{s\,\mathbb{T}} \partial_\theta\, v = 0\,. \tag{7}$$

necessarily holds. Thus, if $\int_{\mathbb{T}} \Phi\, d\theta \neq 0$, then v, and hence also f, is multiply-valued in G, see [1].

But since u is locally in G the real part of the analytic function $f = u + i\,v$, we may lift the local definition of f to the simply-connected Riemann surface $R = \{\sigma \in \mathbb{C} : \log s < \mathrm{Re}\,\sigma < 0\}$, by means of the covering mapping $h : R \to G$, $h(\sigma) = e^\sigma$. That is, there is an analytic function $g : R \to \mathbb{C}$ such that locally $f = g \circ h^{-1}$.

Now $g(\sigma + 2\pi i) - g(\sigma)$ is an analytic function in R whose real part is zero, and it is therefore equal to a (purely imaginary) constant, $i\,c$ say. Thus $g(\sigma) - \frac{c}{2\pi}\,\sigma$ is a $2\pi i$-periodic function of σ. We conclude that there is a single-valued analytic function F defined on G such that

$$F(z) = f(z) - \frac{c}{2\pi}\,\log z\,, \tag{8}$$

whence $u(z) = \mathrm{Re}\,F(z) + \frac{c}{2\pi}\,\log|z|$. Now (7) implies that

$$c = \int_0^{2\pi} \Phi(e^{i\theta})\, d\theta\,. \tag{9}$$

Indeed, if u has the standard representation of a harmonic function in an annulus, namely

$$u(re^{i\theta}) = \sum_{k\in\mathbb{Z},k\neq 0} r^k(a_k \cos k\theta + b_k \sin k\theta) + a_0 + b_0 \ln r,$$

then $c = 2\pi b_0$ and v is given by

$$v(re^{i\theta}) = \sum_{k\in\mathbb{Z},k\neq 0} r^k(a_k \sin k\theta - b_k \cos k\theta) + b_0\theta,$$

and

$$F(z) = \sum_{k\in\mathbb{Z},k\neq 0} (a_k - ib_k)z^k + a_0.$$

On examining the Fourier coefficients of the functions involved, we see immediately that, if $u \in W^{m,2}(\partial G)$ for some m, then v, f and F also lie in $W^{m,2}(\partial G)$, and the Hilbert transform is a contraction with respect to each of these norms.

1.4 Weighted Hardy classes of circular domains

Let G be a *circular domain*, that is, a domain consisting of the open unit disc from which a finite number of pairwise disjoint closed discs have been removed:

$$G = \mathbb{D} \setminus \bigcup_{j=1}^{N} (a_j + r_j \overline{\mathbb{D}}), \tag{10}$$

with the obvious inequalities satisfied by the a_j and r_j for $j = 1, \ldots, N$. We write $D_j = a_j + r_j \mathbb{D}$ for $1 \leq j \leq n$. Let Γ denote the boundary of G. We normalize the Lebesgue measure on Γ so that each circle Γ_j composing it is given unit measure.

The Hardy spaces $H^p(G)$ on a circular domain G were defined by Rudin [32] in terms of analytic functions f such that $|f(z)|^p$ has a *harmonic majorant* on G, that is, a real harmonic function $u(z)$ such that $|f(z)|^p \leq u(z)$ on G.

It is also possible to define the Hardy spaces $H^p(\partial G)$ for $1 \leq p < \infty$ as the closure in $L^p(\partial G)$ of the set R_G of rational functions whose poles lie in the complement of $\overline{G}$. This approach, similar to one in [7], was taken in [17]. The spaces $H^p(G)$ and $H^p(\partial G)$ are then isomorphic in a natural way, and so we identify the two spaces.

Below, we stick to the most completely analysed example of the annulus $G = \mathbb{D} \setminus s\mathbb{D}$ for some fixed s, $0 < s < 1$, and to the Hilbert case where $p = 2$. Here again, ∂G is equipped with the Lebesgue measure normalized so that the circles $\mathbb{T}$ and $s\mathbb{T}$ each have unit measure.

The space $H^2(\partial G)$ has a canonical orthonormal basis consisting of the functions

$$e_n(z) := (z^n / \sqrt{1 + s^{2n}})_{n \in \mathbb{Z}},$$

and it can be written as an orthogonal direct sum

$$H^2(\partial G) = H^2(\mathbb{D}) \oplus H_0^2(\mathbb{C} \setminus s\overline{\mathbb{D}}) \tag{11}$$

of elementary Hardy spaces, by taking the closed linear spans of $(e_n)_{n \geq 0}$ and $(e_n)_{n<0}$ respectively. Here $H_0^2(\mathbb{C} \setminus s\overline{\mathbb{D}})$ is the Hardy space of functions analytic on the complement of $s\overline{\mathbb{D}}$, with L^2 boundary values, and vanishing at infinity. It should be noted that a similar decomposition applies to general spaces $H^p(\partial G)$, but the direct sum is no longer orthogonal in the case $p \neq 2$, see [22, Thm 10.12].

Given sequences $(w_n)_{n \in \mathbb{Z}}$ and $(\mu_n)_{n \in \mathbb{Z}}$ of positive numbers, we introduce $H^2_{w,\mu}(\partial G)$ to be the weighted Hardy space of the annulus G, with the norm

$$\|g\|_{H^2_{w,\mu}(\partial G)} = \sum_{n \in \mathbb{Z}} |g_n|^2 \left[w_n + s^{2n} \mu_n \right],$$

for functions $g(z) = \sum_{n \in \mathbb{Z}} g_n z^n$, $z \in G$. Provided that

$$\inf_{n \in \mathbb{Z}} \frac{w_n + s^{2n} \mu_n}{1 + s^{2n}} > 0, \tag{12}$$

and because the sequence of functions $(z^n/\sqrt{1+s^{2n}})_{n\in\mathbb{Z}}$ is an orthonormal basis of $H^2(\partial G)$, the space $H^2_{w,\mu}(\partial G)$ embeds continuously in the unweighted space $H^2(\partial G)$, and thus its elements possess boundary values on $\mathbb{T}$ and $s\mathbb{T}$, as follows. Let $L^2_w(\mathbb{T}) \subset L^2(\mathbb{T})$ and $L^2_\mu(s\mathbb{T}) \subset L^2(s\mathbb{T})$ respectively be the spaces of functions $g = \sum_{n\in\mathbb{Z}} g_n\, z^n$ such that $\|g\|^2_{L^2_w(\mathbb{T})} = \sum_{n\in\mathbb{Z}} |g_n|^2\, w_n < \infty$ and $\|g\|^2_{L^2_\mu(s\mathbb{T})} = \sum_{n\in\mathbb{Z}} |g_n|^2\, s^{2n}\, \mu_n < \infty$, respectively. Functions belonging to $H^2_{w,\mu}(\partial G)$ thus admit traces on $\mathbb{T}$ and $s\mathbb{T}$ that belong to $L^2_w(\mathbb{T})$ and $L^2_\mu(s\mathbb{T})$, respectively.

More generally, we have a continuous embedding from $H^2_{w,\mu}(\partial G)$ into $H^2_{w',\mu'}(\partial G)$ if and only if

$$\inf_{n\in\mathbb{Z}} \frac{w_n + s^{2n}\mu_n}{w'_n + s^{2n}\mu'_n} > 0. \tag{13}$$

We write $L^2_{w,\mu}(\partial G) \subset L^2(\partial G)$ for the space of those functions defined on ∂G such that their restrictions to $\mathbb{T}$ and $s\mathbb{T}$ lie in $L^2_w(\mathbb{T})$ and $L^2_\mu(s\mathbb{T})$, respectively. Assumption (13) is also necessary and sufficient to ensure that $L^2_{w,\mu}(\partial G)$ is continuously embedded into $L^2_{w',\mu'}(\partial G)'$.

We write $P_{L^2_w(\mathbb{T})}\, g = \chi_\mathbb{T}\, g$ for the function in $L^2_{w,\mu}(\partial G)$ that coincides with g on $\mathbb{T}$ and vanishes on $s\mathbb{T}$. The definition of $P_{L^2_\mu(s\mathbb{T})}$ is analogous.

For $m \geq 1$, introduce $H^{m,2}(\partial G) = H^2(\partial G) \cap W^{m,2}(\partial G)$, the Hardy–Sobolev space of the annulus G, with the $W^{m,2}(\partial G)$ norm:

$$\|g\|_{H^{n,2}(\partial G)} = \|g\|_{W^{n,2}(\partial G)} = \sum_{n\in\mathbb{Z}} |g_n|^2 \left[w_{m,n} + \mu_{m,n} s^{2n}\right],$$

for functions $g \in H^{m,2}(\partial G)$, $g(z) = \sum_{n\in\mathbb{Z}} g_n\, z^n$, $z \in G$, and

$$\begin{cases} w_{m,n} = 1 + n^2 + n^2\,(n-1)^2 + \cdots + n^2\,(n-1)^2 \cdots (n-m+1)^2\,, \\ \mu_{m,n} = 1 + n^2\, s^{-2} + \cdots + n^2\,(n-1)^2 \cdots (n-m+1)^2\, s^{-2m}\,. \end{cases} \tag{14}$$

For consistency of notation, we shall also write $H^{0,2}(\partial G)$ for $H^2(G)$ and $W^{0,2}(\partial G)$ for $L^2(\partial G)$.

1.5 Back to Equations (1) and (2)

Related to the direct problem of finding the solution u and its trace u_b on $\mathbb{T}$ when Φ and φ are given in (1), (2), we have the following result:

Lemma 4 *Let $\Phi \in L^2(\mathbb{T})$ and assume that $\varphi \in A^{(0)}$. There exists a function $f \in H^{1,2}(\partial G)$ such that the solution u to* (1), (2) *satisfies $u = \operatorname{Re} f$ in $\overline{G}$.*

Moreover, there exists a single–valued function $F \in H^{1,2}(\partial G)$ such that $f = F + c \log z$ in $\overline{G}$, for c defined by (9).

More generally, let $n \geq 0$, $\Phi \in W^{n,2}(\mathbb{T})$, $\varphi \in A^{(n)}$. Then $u = \operatorname{Re} f$, for some $f \in H^{n+1,2}(\partial G)$.

The proof of this assertion is contained in Section 1.3 for $n = 0$ and may be established as a corollary for $n \geq 1$.

2 Analytic extensions on circular domains

Let $f_0 \in L^2(\mathbb{T})$, recall that $H^2(\partial G) \cap W^{1,2}(\partial G) = H^{1,2}(\partial G)$, and consider the following mapping:

$$\begin{array}{rcl} \mu : H^{1,2}(\partial G) & \to & \mathbb{R} \\ \xi & \mapsto & \|\xi - f_0\|_{L^2(\mathbb{T})}. \end{array}$$

If f_0 is the trace on $\mathbb{T}$ of a function f in $H^{1,2}(\partial G)$ then μ attains its minimum, namely 0, at $\xi = f$, and this minimum is unique. For if $\xi = h_0$ is another such, then setting $k = h_0 - f \in H^{1,2}(\partial G)$, we have $k = 0$ on $\mathbb{T}$, and we deduce that $k = 0$ on G.

However, the numerical errors inherent in practical experiments will normally prevent f_0 from being exactly the trace of a function in $H^{1,2}(\partial G)$.

Proposition 5 *The space $P_{L^2(\mathbb{T})}H^{1,2}(\partial G)$ is dense in $L^2(\mathbb{T})$.*

Proof: This follows immediately since the functions $z \mapsto z^n$ lie in $H^{1,2}(\partial G)$, and their restrictions to $\mathbb{T}$ form an orthonormal basis of $L^2(\mathbb{T})$. ■

Remark 6 The following generalization of Proposition 5 also holds, and is proved in a similar manner:

if (w_n), (μ_n), (w'_n), and (μ'_n), for $n \in \mathbb{Z}$, are sequences that satisfy (13), then the space $P_{L^2_{w'}(\mathbb{T})}H_{w,\mu}(\partial G)$ is dense in $L^2_{w'}(\mathbb{T})$, while $P_{L^2_w(\mathbb{T})}H_{w,\mu}(\partial G)$ is dense in $L^2_w(\mathbb{T})$.

Proposition 7 ([30]) *Let $f \in L^2(\mathbb{T})$, let $(g_n)_n$ be a sequence in $H^{1,2}(\partial G)$ converging to f in $L^2(\mathbb{T})$. If $f \notin P_{L^2(\mathbb{T})}H^{1,2}(\partial G)$, then $\lim_{n\to\infty} \|g_n\|_{L^2(s\mathbb{T})} = +\infty$.*

Proof: Suppose that $\lim_{n\to\infty} \|g_n\|_{L^2(s\mathbb{T})} \neq +\infty$. Since $(g_n)_n$ converges in $L^2(\mathbb{T})$, it is bounded in $H^{1,2}(\partial G)$, and so there is a subsequence (g_{n_k}), converging weakly to some g in $H^{1,2}(\partial G)$. In particular, $g_{n_k|\mathbb{T}}$ converges weakly to $g_{|\mathbb{T}}$; since $g_{n_k|\mathbb{T}}$ converges strongly to f in $L^2(\mathbb{T})$, we deduce that $f = g_{|\mathbb{T}}$. ■

Thus the preceding minimization problem is unstable, since if f_0 is not the trace on $\mathbb{T}$ of a function from $H^{1,2}(\partial G)$, then by the density result there is a sequence $(g_n)_n$ converging to g_0 in $L^2(\mathbb{T})$, but with $\lim_{n\to\infty} \|g_n\|_{L^2(s\mathbb{T})} = +\infty$. To prevent this unstable behaviour, we introduce a constraint M in the minimization problem for μ, and we shall look for an approximant g_0 in $H^{1,2}(\partial G)$.

This will be the approach of the inverse problem (1), (2), since boundary measures are subject to noise and errors and the function f_0 given by (5) will not truly be analytic.

2.1 Constrained approximation on circular domains

The standard bounded extremal problem for $H^2(\partial G)$ is the following. Let I be a measurable subset of ∂G, and let $J = \partial G \setminus I$. Suppose that I and J are sets of uniqueness for $H^2(\partial G)$ in the sense that no nontrivial function in $H^2(\partial G)$ can be identically zero on either set (a common choice is to take I to be one component of ∂G, or else a closed subarc contained in a single component). Note that if $f \in H^2(\partial G)$, then the restriction of f to I lies in $L^2(I)$.

The problem below is formulated quite generally in weighted Hardy spaces of the annulus, but in general no simple description of the restriction of $H^2_{w,\mu}(\partial G)$ to I is available, unless $I = \mathbb{T}$, or the space is a Hardy–Sobolev space, cases which we analyse in detail.

Problem 8 *Let $f_0 \in L^2_{w,\mu}(I) \setminus P_{L^2_{w,\mu}(I)} H^2_{w,\mu}(\partial G)$, $f_1 \in L^2_{w,\mu}(J)$ and $M > 0$. Find a function $g_0 \in H^2_{w,\mu}(\partial G)$ such that $\|g_0 - f_1\|_{L^2_{w,\mu}(J)} \leq M$ and*

$$\|f_0 - g_0\|_{L^2_{w,\mu}(I)} = \inf\{\|f_0 - g\|_{L^2_{w,\mu}(I))} : \ g \in H^2_{w,\mu}(\partial G)\,, \ \|g - f_1\|_{L^2_{w,\mu}(J)} \leq M\}. \tag{15}$$

In the hilbertian framework, such bounded extremal problems have been considered in [4, 9, 30, 16, 18, 33], even for more general Hilbert spaces, criteria and constraints.

We shall require the solution to the bounded extremal problem for $H^2_{w,\mu}(\partial G)$, where $\partial G = \mathbb{T} \cup s\mathbb{T}$ and $I = \mathbb{T}$, which can expressed as follows (see Remark 12).

Theorem 9 *Let $(w_n)_{n\in\mathbb{Z}}$ and $(\mu_n)_{n\in\mathbb{Z}}$ be two sequences jointly satisfying (12). Let $f_0 \in L^2_w(\mathbb{T}) \setminus P_{L^2_w(\mathbb{T})} H^2_{w,\mu}(\partial G)$, $f_1 \in L^2_\mu(s\mathbb{T})$ and $M > 0$. Then there exists a unique function $g_0 \in H^2_{w,\mu}$ such that $\|g_0 - f_1\|_{L^2_\mu(s\mathbb{T})} \leq M$ and*

$$\|f_0 - g_0\|_{L^2_w(\mathbb{T})} = \inf\{\|f_0 - g\|_{L^2_w(\mathbb{T})} : \ g \in H^2_{w,\mu}\,, \ \|g - f_1\|_{L^2_\mu(s\mathbb{T})} \leq M\}. \tag{16}$$

Indeed, if $f_0(z) \sim \sum_{n\in\mathbb{Z}} a_n z^n$ for $z \in \mathbb{T}$ and $f_1(z) \sim \sum_{n\in\mathbb{Z}} b_n z^n$ for $z \in s\mathbb{T}$, then

$$g_0(z) = \sum_{n\in\mathbb{Z}} \frac{a_n\, w_n + \alpha b_n \mu_n s^{2n}}{w_n + \alpha \mu_n s^{2n}} z^n, \tag{17}$$

where $\alpha > 0$ is the unique constant such that

$$\sum_{n\in\mathbb{Z}} \frac{|(a_n - b_n)\, w_n|^2 s^{2n} \mu_n}{(w_n + \alpha \mu_n s^{2n})^2} = M^2. \tag{18}$$

Proof: Observe first that a function belonging to $H^2_{w,\mu}(\partial G)$ that vanishes on $\mathbb{T}$ or on $s\mathbb{T}$ would necessarily vanish everywhere in G. It is then established in [16, 30] that there exists a unique solution g_0 to problem (16) which saturates the norm constraint on $s\,\mathbb{T}$. Moreover, if $P_{H_{w,\mu}}$ denotes the orthogonal projection from $L^2_{w,\mu}(\partial G)$ onto $H^2_{w,\mu}(\partial G)$ and if T is the Toeplitz-like operator on $H^2_{w,\mu}(\partial G)$ defined by $T\,g = P_{H^2_{w,\mu}(\partial G)}\, P_{L^2_\mu(s\mathbb{T})}\, g$, then g_0 is given by

$$(1 + \lambda T)\, g_0 = P_{H^2_{w,\mu}(\partial G)}\, [f_0 + (1+\lambda)\, f_1]\,, \tag{19}$$

for the unique $\lambda > -1$ such that

$$\|g_0 - f_1\|_{L^2_\mu(s\mathbb{T})} = M\,. \tag{20}$$

It can be checked that, if $f \in L^2_{w,\mu}(\partial G)$, $f(z) \sim \sum_{n\in\mathbb{Z}} a_n z^n$ for $z \in \mathbb{T}$ and $f(z) \sim \sum_{n\in\mathbb{Z}} b_n z^n$ for $z \in s\mathbb{T}$, then

$$P_{H^2_{w,\mu}(\partial G)}\, f = \sum_{n\in\mathbb{Z}} \frac{a_n\, w_n + b_n \mu_n s^{2n}}{w_n + \mu_n s^{2n}} z^n\,.$$

On the orthogonal basis $(z^n)_{n\in\mathbb{Z}}$ of $H^2_{w,\mu}(\partial G)$, the operator T can then be written as:

$$T\,g = \sum_{n\in\mathbb{Z}} \frac{g_n \mu_n s^{2n}}{w_n + \mu_n s^{2n}} z^n\,,$$

for functions $g(z) \sim \sum_{n\in\mathbb{Z}} g_n z^n \in H^2_{w,\mu}(\partial G)$. With these expressions, equations (19) and (20) give the desired result, with $\alpha = 1 + \lambda$. ∎

Hardy–Sobolev results follow from Theorem 9. For $m \geq 1$, the sequences $(w_n) = (w_{m,n})_{n\in\mathbb{Z}}$ and $(\mu_n) = (\mu_{m,n})_{n\in\mathbb{Z}}$ defined by (14) satisfy hypothesis (12) and the following results can directly be deduced from Theorem 9.

Corollary 10 *Let $m \geq 1$, $f_0 \in W^{m,2}(\mathbb{T}) \setminus P_{W^{m,2}(\mathbb{T})} H^{m,2}(\partial G)$, $f_1 \in W^{m,2}(s\mathbb{T})$ and $M > 0$. Then there exists a unique function $g_0 \in H^{m,2}(\partial G)$ such that $\|g_0 - f_1\|_{W^{m,2}(s\mathbb{T})} \leq M$ and*

$$\|f_0 - g_0\|_{W^{m,2}(\mathbb{T})} = \inf\{\|f_0 - g\|_{W^{m,2}(\mathbb{T})} :\ g \in H^{m,2}(\partial G)\,,\ \|g - f_1\|_{W^{m,2}(s\mathbb{T})} \leq M\}. \tag{21}$$

Indeed, if $f_0(z) \sim \sum_{n\in\mathbb{Z}} a_n z^n$ for $z \in \mathbb{T}$ and $f_1(z) \sim \sum_{n\in\mathbb{Z}} b_n z^n$ for $z \in s\mathbb{T}$, then g_0 is determined by equations (17), (18), with $w_n = w_{m,n}$ and $\mu_n = \mu_{m,n}$ defined by (14).

Remark 11 The case $m = 0$ of corollary 10, which corresponds to the unweighted case $w_n = \mu_n = 1$ of Theorem 9, was given in [17] with $f_1 = 0$, that is, with $b_n = 0$ for all n, The full $H^2(\partial G)$ result was established in [33, 34], with further analysis, as an application of the results of [30].

In the case $m = 1$ of $H^{1,2}(\partial G)$, where $w_{1,n} = 1 + n^2$ and $\mu_{1,n} = 1 + n^2\, s^{-2}$, this gives precisely

$$g_0(z) = \sum_{n\in\mathbb{Z}} \frac{a_n\,(1+n^2) + \alpha b_n\,(s^{2n} + n^2\, s^{2n-2})}{1 + n^2 + \alpha(s^{2n} + n^2\, s^{2n-2})} z^n,$$

where $\alpha > 0$ is the unique constant such that

$$\sum_{n\in\mathbb{Z}} \frac{|(a_n - b_n)\,(1+n^2)|^2\,(s^{2n} + n^2\, s^{2n-2})}{[1 + n^2 + \alpha(s^{2n} + n^2\, s^{2n-2})]^2} = M^2.$$

Remark 12 One important special case of Problem 8 is when I is a nontrivial interval contained in $\mathbb{T}$. In this case I is certainly a set of uniqueness for $H^2(\partial G)$ and hence also for $H^{m,2}(\partial G)$: one way to see this is by a conformal mapping argument that shows that any $H^2(\partial G)$ function vanishing on I will vanish on a suitable simply-connected subdomain of G, and hence be identically zero. Also, an easy argument based on Runge's theorem, as in the analogous problem for the disc (cf. [10]), shows that the restriction of $H^{m,2}(\partial G)$ to I is dense in $W^{m,2}(I)$. Standard arguments as in [30] now show that the problem has a unique solution. Its computation will be handled in [26] with other constructive issues.

Remark 13 An important consideration here is the continuity of the solution to Problem 8. This can be deduced from a general abstract formulation given [18, 29], which we now sketch briefly.

Let $\mathcal{H}$, $\mathcal{J}$ and $\mathcal{K}$ be Hilbert spaces and $A : \mathcal{H} \to \mathcal{J}$, $B : \mathcal{H} \to \mathcal{K}$ bounded linear operators such that A has dense range and there is a constant $\eta > 0$ with $\|Af\|^2 + \|Bf\|^2 \geq \eta \|f\|^2$ for all $f \in \mathcal{H}$. Given $f_0 \in \mathcal{J} \setminus \operatorname{Im} A$, $f_1 \in \mathcal{K}$ and $M > 0$, the problem is to find $g_0 \in \mathcal{C} = \{g \in \mathcal{H} : \|Bg - f_1\| \leq M\}$ to minimize $\|Ag - f_0\|$ over $g \in \mathcal{C}$.

In our situation, $\mathcal{H} = H^2_{w,\mu}(\partial G)$ and A and B are the restriction mappings into $\mathcal{J} = L^2_{w,\mu}(I)$ and $\mathcal{K} = L^2_{w,\mu}(J)$ respectively. It is known from [18] that such a problem has a unique solution $g = g_0$ and from [29] that g_0 depends continuously on f_0 and M, if f_1 is fixed.

Thus in our situation the solution to Problem 8 does depend continuously on the data f_0.

2.2 Norm estimations in weighted Hardy spaces of circular domains

We now establish $L^2_{w',\mu'}$ estimates for functions in rather general weighted Hardy classes $H^2_{w,\mu}(\partial G)$. These results are strongly linked to those of [11, 13], which hold in Hardy–Sobolev spaces of the unit disc, the estimates concerning the norm on subsets of the connected boundary $\mathbb{T}$. In the present case, estimates are obtained in Hardy spaces of circular domains, on one of the two components of the boundary ∂G.

Let $B^{m,2}$ denote the unit ball of $H^{m,2}(\partial G)$, and $B^2_{w,\mu}$ the unit ball of $H^2_{w,\mu}(\partial G)$.

Let (w_n), (μ_n), and (w'_n), (μ'_n) be sequences of positive numbers: in order to guarantee boundary values of $H^2_{w',\mu'}(G)$ functions we shall suppose that (w'_n), (μ'_n) satisfy (12); recall that in this case, $H^2_{w',\mu'}(\partial G) \subset H^2(\partial G)$. We shall also assume that

$$\frac{s^{2n}\mu'_n}{w_n + s^{2n}\mu_n} \leq \delta(n) \quad \text{for } n < 0, \quad \text{where } \delta(n) \text{ decreases to 0 as} \quad n \to -\infty, \tag{22}$$

together with

$$\sup_{n\in\mathbb{Z}} \frac{\mu'_n}{w'_n} \leq \varrho\,, \text{ for some constant } \varrho > 0\,. \tag{23}$$

The following result uses the one-sided condition in (22) in order to deduce results about a function's behaviour on $s\mathbb{T}$ from its behaviour on $\mathbb{T}$. This may be contrasted with a later result, Theorem 17, where we use the two-sided conditions (12) and (13) in order to deduce the convergence to zero on ∂G of a sequence of functions from convergence (in another norm) on a subset of ∂G.

Theorem 14 *Assume that hypotheses (12), (22) and (23) are satisfied for sequences (w_n), (μ_n) and (w'_n), (μ'_n) of positive numbers. Let $g \in H^2_{w,\mu}(\partial G)$ be such that $g \in B^2_{w,\mu}$ and $\|g\|_{L^2_{w'}(\mathbb{T})} \leq \varepsilon$ for some $\varepsilon > 0$. Then*

$$\|g\|_{L^2_{\mu'}(s\mathbb{T})} \leq \left(\delta\left(-1 - \left\lfloor\left|\frac{\log\varepsilon}{2\log s}\right|\right\rfloor\right) + \varrho\varepsilon\right)^{1/2} \leq \left(\delta\left(-\left|\frac{\log\varepsilon}{2\log s}\right|\right) + \varrho\varepsilon\right)^{1/2}.$$

In particular, if there are constants $c > 0$ and $\alpha > 0$ such that $\delta(n) \leq c|n|^{-\alpha}$, then there exists a constant $C > 0$ such that

$$\|g\|_{L^2_{\mu'}(s\mathbb{T})} \leq \frac{C}{|\log\varepsilon|^{\alpha/2}}.$$

Proof: we estimate the quantity

$$\begin{aligned}\|g\|^2_{L^2_{\mu'}(s\mathbb{T})} &= \sum_{n\leq -N} s^{2n}\mu'_n|g_n|^2 + \sum_{n=-N+1}^{\infty} s^{2n}\mu'_n|g_n|^2 \\ &= \sigma_1 + \sigma_2, \quad \text{say.}\end{aligned}$$

Because $g \in B^2_{w,\mu}$, we see that

$$\sum_{n\leq -N} |g_n|^2\,(w_n + s^{2n}\,\mu_n) \leq 1\,.$$

Hence $\sigma_1 \leq \sup_{n\leq -N}\delta(n) = \delta(-N)$. Moreover, $\sum_{n\in\mathbb{Z}} w'_n|g_n|^2 \leq \varepsilon^2$, and hence $\sigma_2 \leq s^{-2(N-1)}\varrho\varepsilon^2$. Choosing $N = 1 + \left\lfloor\left|\frac{\log\varepsilon}{2\log s}\right|\right\rfloor$, we have

$$\|g\|^2_{L^2_{\mu}(s\mathbb{T})} \leq \delta(-N) + \varrho\varepsilon,$$

and the result follows. ■

Corollary 15 *Let m and k be integers with $m > k \geq 0$. Then there exists a constant C, depending only on m, k and s, such that whenever $g \in B^{m,2}$ with $\|g\|_{W^{k,2}(\mathbb{T})} \leq \varepsilon$ for some $\varepsilon > 0$, we have*

$$\|g\|_{W^{k,2}(s\mathbb{T})} \leq \frac{C}{|\log\varepsilon|^{m-k}}.$$

Proof: This follows from Theorem 14, on taking the weights $w_n = w_{m,n}$, $\mu_n = \mu_{m,n}$ and $w'_n = w_{k,n}$, $\mu'_n = \mu_{k,n}$. We then have Condition (12) for both (w_n, μ_n) and (w'_n, μ'_n) because all the weights are greater than or equal to 1. Condition (22) holds with $\delta(n)$ of order $|n|^{-2(m-k)}$ for $n < 0$, since μ_n grows as $|n|^{2m}$ and μ'_n grows as $|n|^{2k}$ as $n \to -\infty$. Finally, we also have (23) directly from the definition of the weights given in (14), with $\varrho = s^{-2k}$. ∎

Remark 16 The estimate of Corollary 15 can be shown to be sharp, by considering for instance functions of the form $g(z) = \delta z^p$, where $\delta > 0$ and p is a negative integer. Given $\varepsilon > 0$, choose p as large in absolute value as possible such that

$$\|g\|_{W^{m,2}(\mathbb{T})}/\|g\|_{W^{k,2}(\mathbb{T})} \leq 1/\varepsilon,$$

and then choose δ such that $\|g\|_{W^{k,2}(\mathbb{T})} = \varepsilon$, and hence $g \in B^{m,2}$. It is easy to see that p is asymptotic to $-\log\varepsilon/\log s$, as $\varepsilon \to 0$. But $\|g\|_{W^{k,2}(s\mathbb{T})}$ is now asymptotic to $c_1\delta|p|^k s^p$, which is asymptotic to $c_2|\log\varepsilon|^{k-m}$, where c_1 and c_2 are constants depending on m, k and s, but not p, δ or ε. The result follows.

Whenever the norm of the $H^2_{w,\mu}(\partial G)$ function is known to be "small" on only a subset I of ∂G, we can still conclude that its norm on the whole boundary ∂G remains small (in a certain sense to be made precise). The following result requires a stronger assumption than (13), but it is satisfied in the situation we are presenting.

Theorem 17 *Let (w'_n), (μ'_n) be weight sequences satisfying (12), and suppose additionally that (w_n), (μ_n) is another weight sequence such that*

$$\sup_{|n|\geq N} \frac{w'_n + s^{2n}\mu'_n}{w_n + s^{2n}\mu_n} \to 0$$

as $N \to \infty$. Let $I \subset \partial G$ be a subset with strictly positive measure, and suppose that (g_p) is a sequence of functions in $B^2_{w,\mu} \subset H^2_{w,\mu}(\partial G)$ such that $\|g_p\|_{L^2(I)} \to 0$. Then $\|g_p\|_{H^2_{w',\mu'}(\partial G)} \to 0$.

Proof: We claim that $B^2_{w,\mu}$ is a compact subset of $H^2_{w',\mu'}(\partial G)$. It is closed, since if (g_p) is a sequence in $B^2_{w,\mu}$ which converges to g in $H^2_{w',\mu'}(\partial G)$, then for every $N > 0$ the Fourier coefficients $g_{p,n}$ of g_p satisfy

$$\sum_{n=-N}^{N} |g_{p,n}|^2(w_n + s^{2n}\mu_n) \leq 1,$$

and thus the same holds for g. Now taking the limit as $N \to \infty$ shows that $g \in B^2_{w,\mu}$, which is therefore closed. It is also a bounded subset and for every $g \in B^2_{w,\mu}$ we have that

$$\sum_{|n|\geq N} |g_n|^2(w'_n + s^{2n}\mu'_n) \leq \sup_{|n|\geq N} \frac{w'_n + s^{2n}\mu'_n}{w_n + s^{2n}\mu_n} \to 0$$

as $N \to \infty$ (uniformly for $g \in B^2_{w,\mu}$). We deduce easily that $B^2_{w,\mu}$ is a totally bounded subset of $H^2_{w',\mu'}(\partial G)$, and hence compact (cf. [21, ch. IV, Ex. 13] for a compactness criterion in ℓ^2, which is easily adapted here).

Now, let (g_p) satisfy the assumptions of the proposition. Either $\|g_p\|_{H^2_{w',\mu'}(\partial G)} \to 0$, or, after extracting a subsequence and relabelling, we may suppose that (g_p) converges in $H^2_{w',\mu'}(\partial G)$ norm to some function g lying in $B^2_{w,\mu}$, a compact set; however, g necessarily vanishes on I. Now I is a uniqueness set in $H^2(\partial G)$ (this can be seen using the fact that either $I \cap \mathbb{T}$ or $I \cap s\mathbb{T}$ has positive measure, and g also lies in $H^2(E)$ for some suitable simply-connected domain $E \subset G$ with a smooth boundary). Hence g vanishes identically, and this is a contradiction. ∎

Corollary 18 *Let m and k be integers with $m > k \geq 0$, and let $I \subset \partial G$ be a compact subset with nonempty interior. Let (g_p) be a sequence of functions in $B^{m,2} \subset H^{m,2}(\partial G)$ such that $\|g_p\|_{L^2(I)} \to 0$. Then $\|g_p\|_{H^{k,2}(\partial G)} \to 0$.*

Proof: It is easily verified that the weights $w_n = w_{m,n}$, $\mu_n = \mu_{m,n}$ and $w'_n = w_{k,n}$, $\mu'_n = \mu_{k,n}$ satisfy the conditions of Theorem 17. ∎

3 Stability results and error estimates

We return to the Cauchy problem (1), (2) in the annulus G, and we are now in a position to state our main results.

Theorem 19 *Let $\Phi_1, \Phi_2 \in W^{1,2}(\mathbb{T})$, $\Phi_1, \Phi_2 \geq 0$ and $\Phi_1, \Phi_2 \not\equiv 0$, and $\varphi_1, \varphi_2 \in A^{(1)}$. Let u_1, u_2 be the associated solutions to (1), (2), and assume that:*

$$\| u_1 - u_2 \|_{L^2(\mathbb{T})} \leq \varepsilon \,, \quad \| \Phi_1 - \Phi_2 \|_{\mathbb{L}^2(\mathbb{T})} \leq \varepsilon \,, \tag{24}$$

for some $\varepsilon > 0$. Then, there exists a constant $K = K(s, A^{(1)}) > 0$ such that the following estimate holds:

$$\| \varphi_1 - \varphi_2 \|_{\mathbb{L}^2(s\mathbb{T})} \leq \frac{K}{|\log \varepsilon|}. \tag{25}$$

Proof: We define $u = u_1 - u_2$, $\Phi = \Phi_1 - \Phi_2$, and $\varphi = \varphi_1 - \varphi_2$. On $s\mathbb{T}$, we have that $\partial_n u_i + \varphi_i u_i = 0$, whence $\partial_n u + \varphi u_1 + \varphi_2 u = 0$ and

$$\varphi u_1 = -\partial_n u - \varphi_2 u \qquad \text{on } s\mathbb{T}\,.$$

By hypothesis, and from (3) in Theorem 1, we get

$$\| \varphi \|_{\mathbb{L}^2(s\mathbb{T})} \leq \frac{\sqrt{2}}{m} \max(1, C_s) \, \| u \|_{W^{1,2}(s\mathbb{T})}\,. \tag{26}$$

Introduce now as in (5) the functions f_i analytic in G such that $u_i = \mathrm{Re}\, f_i$. Together with the regularity results of Theorem 1, Lemma 4 is to the effect that $f_j \in H^{2,2}(\partial G)$. The following Gagliardo–Nirenberg interpolation inequality [12, Chap. VIII]:

$$\| D^1 u \|^2_{L^2(\mathbb{T})} = \| u' \|^2_{L^2(\mathbb{T})} \leq c \, \| u \|_{W^{2,2}(\mathbb{T})} \| u \|_{L^2(\mathbb{T})} , \tag{27}$$

for some $c \geq 1$, holds for all $u \in W^{2,2}(\mathbb{T})$, whence

$$\| u \|^2_{W^{1,2}(\mathbb{T})} \leq (c\, \kappa + \| u \|_{L^2(\mathbb{T})}) \; \| u \|_{L^2(\mathbb{T})} \leq (c\, \kappa + \| u \|_{L^2(\mathbb{T})}) \; \| u \|_{L^2(\mathbb{T})} ,$$

from (4) and for some $\kappa \geq 1$. Next, if $f = f_1 - f_2$, then

$$\| f \|^2_{W^{1,2}(\mathbb{T})} \leq \| u \|^2_{W^{1,2}(\mathbb{T})} + \| \Phi \|^2_{L^2(\mathbb{T})} \leq (c\, \kappa + 2)\, \varepsilon ,$$

by hypothesis, as soon as $\varepsilon < 1$.

Further, from (4) in Theorem 1 and Lemma 4, there exists $\kappa' \geq \kappa$ (depending on s and the class $A^{(1)}$) such that $f/\kappa' \in B^{2,2}$. Now, let $k = \max(c\, \kappa + 2, \kappa') \geq 1$. We have:

$$\| f/k \|_{W^{1,2}(\mathbb{T})} \leq \| f/\kappa \|_{W^{1,2}(\mathbb{T})} \leq \frac{\varepsilon^{1/2}}{\kappa} \leq \varepsilon^{1/2} ,$$

while $f/k \in B^{2,2}$. In view of Corollary 15, this leads to

$$\| f/k \|_{W^{1,2}(s\mathbb{T})} \leq \frac{C}{|\log \varepsilon^{1/2}|} = \frac{2\, C}{|\log \varepsilon|} ,$$

for some $C > 0$ (depending on s). Finally, since $\| u \|_{W^{1,2}(s\mathbb{T})} \leq \| f \|_{W^{1,2}(s\mathbb{T})}$, we conclude from (26) that:

$$\| \varphi \|_{\mathbb{L}^2(s\mathbb{T})} \leq \frac{2\,\sqrt{2}\, C\, k\, \max(1, C_s)}{m\, |\log \varepsilon|} .$$

■

In the uniform norm, we have the following:

Corollary 20 *Let $\Phi_1, \Phi_2 \in W^{2,2}(\mathbb{T})$, $\Phi_1, \Phi_2 \geq 0$ and $\Phi_1, \Phi_2 \not\equiv 0$, and $\varphi_1, \varphi_2 \in A^{(2)}$. Let u_1, u_2 be the associated solutions to* (1), (2), *and assume that:*

$$\| u_1 - u_2 \|_{L^\infty(\mathbb{T})} \leq \varepsilon \, , \;\; \| \Phi_1 - \Phi_2 \|_{L^\infty(\mathbb{T})} \leq \varepsilon \, , \tag{28}$$

for some $\varepsilon > 0$. Then, there exists a constant $K = K(s, A^{(2)}) > 0$ such that the following estimate holds:

$$\| \varphi_1 - \varphi_2 \|_{\mathbb{L}^\infty(s\mathbb{T})} \leq \frac{K}{|\log \varepsilon|} . \tag{29}$$

Proof: With the notations already used in the proof of Theorem 19, we have by hypothesis and from (3) that:

$$\| \varphi \|_{\mathbb{L}^\infty(s\mathbb{T})} \leq \frac{1}{m} \, \max(1, C_s) \, \| u \|_{W^{1,\infty}(s\mathbb{T})} . \tag{30}$$

Next, from Lemma 4, $u = \mathrm{Re}\, f$ for a function $f \in H^{3,2}(\partial G)$ and there exists $\kappa' \geq 1$ (depending on s and the class $A^{(2)}$) such that $f/\kappa' \in B^{3,2}$. Further,

$$\| u \|_{W^{1,\infty}(s\mathbb{T})} \leq \| u \|_{W^{2,2}(s\mathbb{T})} \leq \| f \|_{W^{2,2}(s\mathbb{T})} .$$

Here, (27) will be to the effect that

$$\| f \|^2_{W^{2,2}(\mathbb{T})} \leq \kappa' \, (c+1) \, \| f \|_{W^{1,2}(\mathbb{T})} .$$

Now, applying once again (4) and (27) to u, we get

$$\| f \|^2_{W^{1,2}(\mathbb{T})} = \| u \|^2_{W^{1,2}(\mathbb{T})} + \| \Phi \|^2_{L^2(\mathbb{T})} \leq \kappa \, (c+1) \, \| u \|_{L^2(\mathbb{T})} + \| \Phi \|^2_{L^2(\mathbb{T})}$$

$$\leq \kappa \, (c+1) \, \| u \|_{L^\infty(\mathbb{T})} + \| \Phi \|^2_{L^\infty(\mathbb{T})} \leq \kappa \, (c+2) \, \varepsilon \, ,$$

by hypothesis. Hence,

$$\| f \|^2_{W^{2,2}(\mathbb{T})} \leq \kappa' \sqrt{\kappa} \, (c+1) \sqrt{c+2} \, \varepsilon^{1/2} \leq k^2 \, \varepsilon^{1/2} \, .$$

for $k = \kappa' \, (c+2)$ for instance. We are then in position to apply Corollary 15 to f/k and:

$$\| f \|_{W^{2,2}(s\mathbb{T})} \leq \frac{4 \, C \, k}{|\log \varepsilon|} \, .$$

Altogether and referring back to (30), this finally leads to:

$$\| \varphi \|_{\mathbb{L}^\infty(s\mathbb{T})} \leq \frac{4 \, C \, k \, \max(1, C_s)}{m \, |\log \varepsilon|} \, .$$

■

Remark 21 If $\mathcal{G}$ is a domain equivalent to an annulus G by a conformal transformation with a $C^{2,\beta}$ extension, $0 < \beta < 1$, that maps $\partial\mathcal{G}$ to ∂G, then, as in the case of simply-connected domains D and the disc $\mathbb{D}$, studied in [13], the inverse Robin problem for $\mathcal{G}$ can be re-expressed as an inverse Robin problem in G. It is clear that stability estimates analogous to those in Theorem 19 and Corollary 20 can be derived in this more general setting too. This holds if $\mathcal{G}$ is doubly-connected with a $C^{2,\beta}$ boundary made of two Jordan closed curves, one of them being strictly contained in the interior of the bounded domain delimited by the other.

Remark 22 Observe that the two proofs above also contain estimates of the errors on inner boundary data $\|u_1 - u_2\|_{W^{1,p}(s\mathbb{T})}$ and $\|\partial_n u_1 - \partial_n u_2\|_{L^p(s\mathbb{T})}$ for $p = 2, \infty$. Further, they can easily be extended to higher order Sobolev spaces.

4 Preliminary numerical results: Cauchy problems in an annulus

Computational issues will be the principal topic of [26]. However, we present in this section some preliminary numerical results – linked to practical situations taken from the engineering sciences – in order to illustrate the constructive aspects of the approximation scheme described in Section 2.1 for data completion purposes, related to Laplace equations in doubly connected domains, in 2D (or 3D axi-symmetrical) situations. These algorithms are numerically quite efficient for solving inverse problems of data extension type, since they do not require iterative resolution of associated direct problems.

4.1 Reconstruction of missing boundary data

Our first application is devoted to the reconstruction of temperature field in a pipeline of an infinite length. This application may arise in several industrial processes. An example taken from fluid mechanics consists in reconstructing the heat data at the internal wall of a pipeline. Such data may be required for the simulation of the heat transfer taking place in a fluid flowing within that pipeline. The knowledge of this temperature is necessary for controlling the safety of the material: stratified inner fluid may generate mechanical stresses, which may cause damage such as cracks. From the experimental viewpoint, thermocouples are located at the external boundary of the pipe and the heat exchange conditions with the environment are known. In the following numerical experiments we assume that the temperature does not depend on the longitudinal coordinate (the z-coordinate) so that we deal with a two-dimensional situation. The cross-sections coincide with $G = \mathbb{D} \setminus s\mathbb{D} \subset \mathbb{R}^2$, the annular domain of radii s and 1, $\mathbb{D}$ being the unit disc.

Figure 1 shows the recovered flux and temperature on $s\mathbb{T}$, $s = 0.6$, for $f_0(z) = 2+1/z-4$, which provides us with the trace on $\mathbb{T}$ of the harmonic function $u_0 = \operatorname{Re} f_0$ together with that of its normal derivative, $\partial_n u_0 = \partial_\theta \operatorname{Im} f_0$. We then compute the solution g_0 to Problem (21) with $m = 1$, $f_1 = 0$, $M \simeq \|f_0\|_{L^2(s\mathbb{T})}$. Observe that the results are pretty good, though the function to be recovered on $s\mathbb{T}$ possesses a singularity in the plane (here at $(x, y) = (4, 0)$).

4.2 Identification of Robin type coefficients

Still in the thermal framework, once the flux and the temperature at the internal (inaccessible) boundary $s\mathbb{T}$, have been computed (as above), we can evaluate the Fourier heat transfer coefficients φ from equation (2). With the above choices, if φ_0 is associated to u_0 and $\partial_n u_0$, and φ_{comp} from $\operatorname{Re} g_0$ and $\partial_n \operatorname{Re} g_0$, we obtain the plots of Figure 2.

Stability properties and error estimates from Theorem 19 and Corollary 20 are illustrated by Figure 3. We took here s, u_0, and $\partial u_0/\partial n$ as above, while u_1 and $\partial u_1/\partial n$ are given by $f_1 = f_0 + \varepsilon\,(2 + 1/z)$ and depend on a parameter $\varepsilon > 0$. The plots in Figure 3 show the

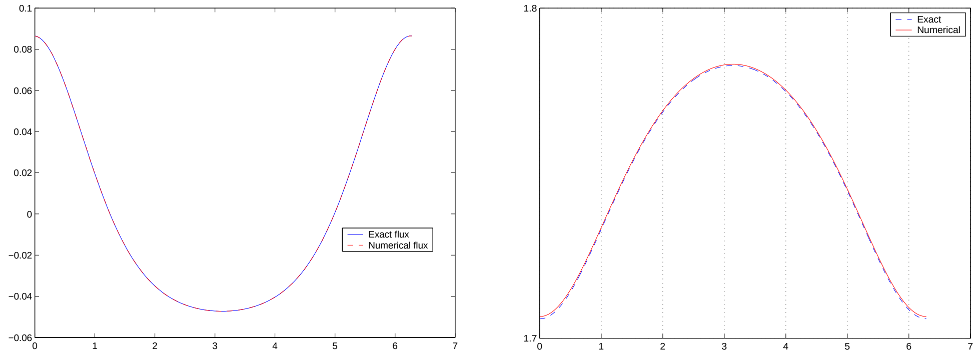

Figure 1: $\partial_n \operatorname{Re} g_0$, $\partial_n u_0$, and $\operatorname{Re} g_0, u_0$ on $s\mathbb{T}$

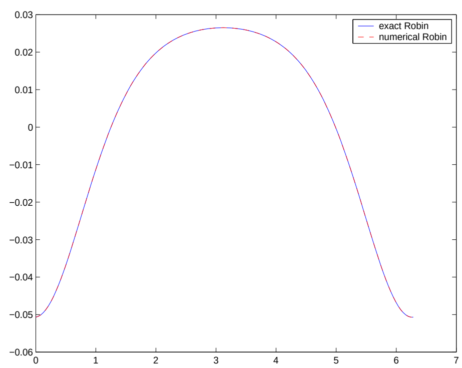

Figure 2: $\varphi_{comp}, \varphi_0$

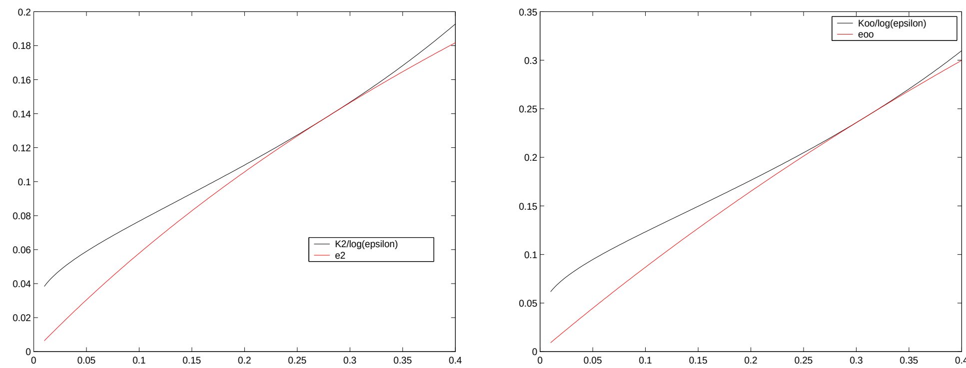

Figure 3: $e_p, K_p/|\log\varepsilon|$, $p = 2, \infty$

behaviour of $e_p = \|\varphi_0 - \varphi_1\|_{L^p(s\mathbb{T})}$ with respect to ε and $K_p/|\log\varepsilon|$ for $K_p = \max_\varepsilon |\log\varepsilon|\, e_p$, for $p = 2, \infty$.

4.3 Source recovery

Our next application to test the performance of the proposed method is motivated by a 2D version of an inverse source problem arising in location of epileptic foci in human brain, the so–called inverse electroencephalography (EEG) problem, [8, 23]. In the so–called spherical model of the to be head, it is assumed to be a ball Ω made up of (at least) 3 disjoint homogeneous connected layers Ω_i (corresponding to scalp, skull and brain), Ω_0 being the inner sphere, with spherical boundaries and constant conductivity $\sigma = \sigma_i$. The steady state electric potential is the solution to

$$\begin{cases} -\nabla.(\sigma\nabla\, u) = \displaystyle\sum_{k=1}^{m} p_k.\nabla\,\delta_{c_k} & \text{in } \Omega, \\ \sigma\partial_n u = \Phi & \text{on } \partial\Omega\,, \ \displaystyle\int_{\partial\Omega}\Phi = 0 \end{cases}$$

where σ is constant in each layer and $c_k \in \Omega_0$, $p_k \in \mathbb{R}^3$. Given measures of u on the outer boundary $\partial\Omega$, the issue is to recover the location of pointwise dipolar sources c_k located in the inner domain Ω_0, and their moments p_k. This involves harmonic data propagation from the outer boundary to the inner one as a preliminary step, after which the singularities are to be recovered. Indeed, on each spherical layer

$$\Delta\, u = 0\,, \text{ in } \Omega_i\,,\ i = 1, 2\,,$$

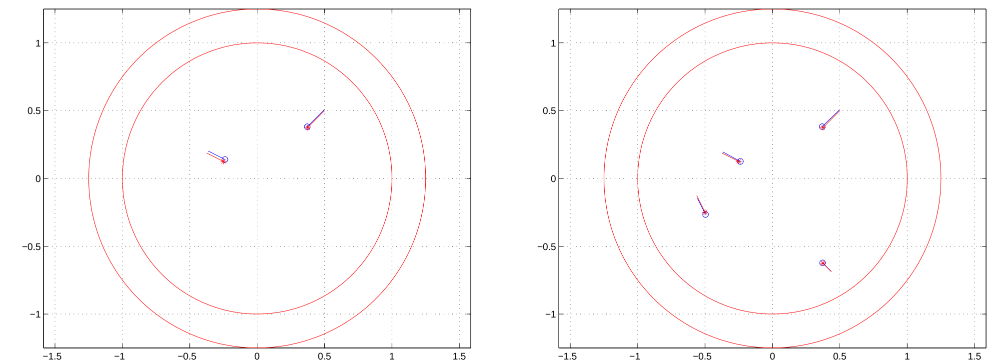

Figure 4: $m = 2$ and 4 dipoles

and propagation condition are required on connected components on the boundaries $\Gamma_0 = \partial\Omega_0$, $\Gamma_i \cup \Gamma_{i-1} = \partial\Omega_i$:

$$\begin{cases} u_{|\Gamma_{i-1}} = u_{|\Gamma_i} \,, \\ \\ \sigma_{i-1}\, \partial_n u_{|\Gamma_{i-1}} = \sigma_i\, \partial_n u_{|\Gamma_i} \,. \end{cases}$$

If we consider the 2D configuration of the above model, which can serve as a discretized version of the 3D one if u depends on the z-coordinate only to the first order, we handle once again the issue of harmonic data propagation in a family of annuli.

Here, we consider two layers, whence a domain $G = R\mathbb{D} \setminus \mathbb{D}$, with $R = 1.25$, $\sigma_0 = 10$, $\sigma_1 = 1$. We first compute using an iterative algorithm from [27] the solution u on the outer boundary $\partial\Omega = R\mathbb{T}$ for various sources locations. We then compute the solution g_0 to Problem (21) with $f_0 = u + i \int \Phi$ on $R\mathbb{T}$, $\Phi(Re^{i\theta}) = \cos\theta$, $m = 1$, $f_1 = 0$, $M \simeq \|f_0\|_{L^2(\mathbb{T})}$.

As a second step, once data are available on the intermediate boundary $s\mathbb{T}$, meromorphic or rational approximation schemes are used in order to approximately locate $\{c_k\}$, as in [8]: see Figure 4.

4.4 Recovery of cracks on circular interfaces

Much work has been done on the very practical problem of recovering interior conductivity defaults of some object from overspecified boundary data. One should expect that sharp results may be obtained by incorporating prior information about the expected geometric features of these inhomogeneities, that we here assume to be cracks. For questions of uniqueness and stability, we refer the reader to [3] and references therein. Identifying cracks located on an interface is a crucial issue in detecting coating defects or delamination in composite material. Coating is a current procedure to combat corrosion. In the present situation, we assume that these interfaces or inner boundaries are circular or spherical.

We consider the steady state heat conduction problem in the annulus G:

$$\begin{cases} \Delta u = 0 & \text{in } G \setminus \sigma, \\ \partial_n u = 0 & \text{on } \sigma \subset \lambda\mathbb{T},\ s < \lambda < 1 \\ \partial_n u = \Phi & \text{on } \partial G, \\ \displaystyle\int_{\partial G} \Phi = 0, & \end{cases}$$

where $\sigma \subset \lambda\mathbb{T}$ corresponds to the unknown co-circular cracks. This time, overdetermined data u_b and Φ are available on the whole $\partial G = s\mathbb{T} \cup \mathbb{T}$ and the key point is to recover σ from these data and the following result. Let $[u]$ denote the jump of u on both sides of a circle in G.

Lemma 23 ([5]) *If $\sigma \subset \lambda\mathbb{T}$ and if $\int_\sigma [u](\lambda e^{i\tau})\, d\tau \neq 0$, then $\sigma = \overline{\{ z \in \lambda\mathbb{T} :\ [u](z) \neq 0 \}}$.*

The above hypothesis concerning the integral of $[u]$ on σ ensures that the flux Φ allows us to identify the cracks or, in other words, ensures that they are not contained in level lines of u.

Assume for the moment that one knows the parameter $\lambda \in (s, 1)$ which defines the "host circle" $\lambda\mathbb{T}$ containing the cracks. We can then split G into two annular subdomains: $G = G_+ \cup G_-$ with $G_+ = \mathbb{D} \setminus \overline{\lambda\mathbb{D}}$ and $G_- = \lambda\mathbb{D} \setminus \overline{s\mathbb{D}}$.
The temperature and the heat flux on the outer boundaries $s\mathbb{T}$, $\mathbb{T}$ are provided by (u_b, Φ), the function u being harmonic now in G_+ and G_-. Hence, (1) holds both in G_+ and G_- with boundary conditions on $\mathbb{T}$ and $s\mathbb{T}$, respectively, and the issue is the reconstruction on the common boundary $\lambda\mathbb{T}$ of the extensions u_+ and u_- of (u_b, Φ). Indeed, this will provide us with the jump $[u] = u_+ - u_-$ on $\lambda\mathbb{T}$, whose support corresponds to σ in view of Lemma 23.

Hence, defining f_+ and f_- as in (5) on $\mathbb{T}$ and $s\ \mathbb{T}$ respectively, we can use the approximation schemes from Section 2.1 in order to compute their best approximants g_+ and g_- in G_+ and G_-. Since $u_\pm \simeq \operatorname{Re} g_\pm$, this allows us to obtain $[u] \simeq g_+ - g_-$ on $\lambda\mathbb{T}$ and to localize σ using Lemma 23.

Preliminary numerical results are shown in Figure 5, which corresponds to the temperature $u_b = \operatorname{Re} f_0$ and to the corresponding flux for $f_0(z) = 2 + z^2$ and $\lambda = 0.85$. The bounded extremal problem (21) is then solved with f_0, $M \simeq \|f_0\|_{L^2(s\mathbb{T})}$, and $m = 0$. In the more realistic case where the host circle $\lambda\mathbb{T}$ is unknown, the above tool can also be used in order to determine the radius λ. The idea here is to introduce a circle $r\mathbb{T}$ for $s < r < 1$ and to run the above approximation process in the two corresponding annular domains $G_{r,+}$ and $G_{r,-}$; if $r < \lambda$, the error in $H^2(G_{r,+})$ approximation will remain large enough (since then, $\sigma \subset G_{r,+}$ and the function f_+ is not the trace of an analytic function there), whence the error in $H^2(G_{r,-})$ from f_- will become small. Of course, if $r > \lambda$, the converse holds, while errors on both sides will be small whenever $r = \lambda$.

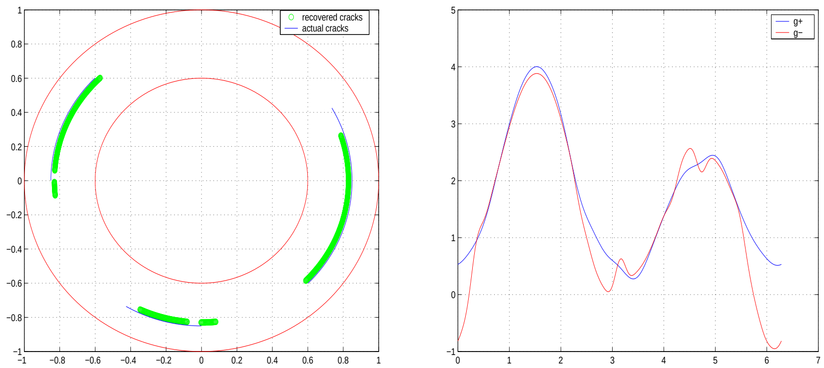

Figure 5: cracks and approximants $g_\pm$ on $\lambda\mathbb{T}$

This step, as well as the localization of σ, could also be performed using the following "reciprocal gap" functional:

$$\int_{\partial\Omega} (\Phi\, h - u_b\, \partial_n h)\; d\tau = \int_\sigma [u] \partial_n h \; d\tau \ ,$$

where the identity above (the Green formula) holds for any function h, harmonic in G. Together with Lemma 23, appropriate choices of test functions h (as real or imaginary parts of z^k, $k \in \mathbb{Z}$) allow one to get both an estimate of λ and a decomposition of $[u]$ on $\lambda\mathbb{T}$. This will be studied in more detail in further work [26].

5 Conclusion

The properties of Section 2.2 hold also for domains conformally equivalent to an annulus by a conformal mapping that is sufficiently smooth on ∂G, since the corresponding Hardy–Sobolev spaces are then mapped to each other with an equivalence of norms; see Remark 21. Some geometrical restrictions on these domains should be added in order to get bounds of the constants arising in the errors estimates.

Note that conformal mappings can also be used for solving geometrical inverse problems and to express them as data recovery ones, see e.g. [28].

Further numerical and computational issues will be considered in the forthcoming work [26].

It is possible to extend virtually all the preceding analysis to domains with more than one hole, the only difficulty being computational, since the orthogonal projections from $L^2(\partial G)$ onto $H^2(\partial G)$ are no longer so simple to write down. One way is to use the reproducing kernels for such spaces, which are given in [7].

Approximation in such domains should also be of use in order to estimate the number of singularities of solutions to Laplace equation and to get an idea of their sizes and locations.

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